

Appendix A

A Tutorial on Magic Numbers for Their Use in Annuities

The “Magic Number,” which goes by the Rule of 72, has been in existence since prior to the year 1514, according to Wikipedia. It is the rule that enables the determination of the approximate number of years it will take money to double, given an effective annual interest rate. The Magic Number is also referred to as the Rule of 69.3 and the rule of 70, rounding up the 69.3. The derivation of this lower magic number is as follows. The future amount, S , of a principal of investment, P , at an effective annual interest rate of $i\%$ is given by

$$S = P(1 + i)^n, \quad (\text{A-1})$$

and

$$(1 + i)^n = \frac{S}{P}. \quad (\text{A-2})$$

If the future amount is some multiple (m) of the investment principal, or P , the properties of logarithms gives

$$n = \frac{\ln(m)}{\ln(1 + i)}. \quad (\text{A-3})$$

Now, for small values of i (the interest rate), the natural logarithm of $(1 + i)$ is approximately equal to i and $m = 2$ for doubling. Then, an equation based on formula A-3 reduces to $n \approx (0.693 / i) \approx (69.3 / i\%)$ and is referred to as the rule of 69.3, after multiplying the numerator and denominator by 100. The numerator is often rounded up to 70, which facilitates interest rates of 7%, 10% and multiples thereof. Wikipedia shows empirical corrections enabling a greater accuracy. To achieve the Rule of 72, the equation is multiplied and divided by i and rearranged as follows (Kellison: *Theory of Interest*, Irwin, 1970).

$$n = \frac{\ln(m)}{i} \times \frac{i}{\ln(1 + i)} \quad (\text{A-4})$$

At an effective annual interest rate of 8%, $i / \ln(1 + i) = 0.08 / \ln(1.08) \approx 1.0395$, and Equation (A-3) becomes

$$n \approx \frac{1.0395 \ln(m)}{i}. \quad (\text{A-5})$$

The Rule of 72 – Doubling

This rule evolves from Equation (A-5) when $m = 2$ and

$$n \approx \frac{1.0395 \times \ln(2)}{0.08} \approx \frac{(1.0395) \times (0.693)}{0.08} \approx \frac{0.7204}{0.08} \approx \frac{72.04}{8} \approx 9.005 \text{ years.}$$

Equation (A-3) gives the exact number of years to double money with an 8% interest rate as 9.007 years.

Then the **Rule of 72** is given by $n \approx \frac{72}{i\%}$.

The Rule of 114 – Tripling

This rule evolves from Equation A-5 when $m = 3$ and

$$n \approx \frac{1.0395 \times \ln(3)}{0.08} \approx \frac{(1.0395) \times (0.10986)}{0.08} \approx \frac{1.1420}{0.08} \approx \frac{114.2}{8} \approx 14.28 \text{ years.}$$

Equation (A-3) gives the exact number of years to triple money with an 8% interest rate as 14.27 years. Then the **Rule of 114** is given by $n \approx 114 / i\%$.

The Rule of 167 – Quintupling

This rule evolves from Equation A-5 when $m = 5$ and

$$n \approx \frac{1.0395 \times \ln(5)}{0.08} \approx \frac{(1.0395) \times (1.6094)}{0.08} \approx \frac{1.673}{0.08} \approx \frac{167.3}{8} \approx 20.91 \text{ years.}$$

Equation (A-3) gives the exact number of years for money to increase 5-fold with an 8% interest rate as 20.91 years. Then the **Rule of 167** is given by $n \approx 167 / i\%$.

Trowbridge used these three “magic numbers” to determine the approximate number of years for multiples of 2, 3 and 5 and how they could be used to determine the approximate number of years to deplete a present value, given a periodic withdrawal or the approximate level annual withdrawal amount to deplete a given present value, given the number of years. (*The Actuary*, American Society of Actuaries, October, 1985)

Multiples of the Magic Numbers

For multiples of these three magic numbers, there is a property of logarithms that states that the logarithm of a product is the sum of the logarithms of the factors that comprise the product. Thus if a multiple of 2, 3, or 5 is separated into the individual factors, the appropriate rules may be used. That is, $\ln(m_1 \times m_2 \times m_3 \times \dots) = \ln(m_1) + \ln(m_2) + \ln(m_3) + \dots$ and the approximate number of years would be determined by

$$n \approx \frac{M(m_1) + M(m_2) + M(m_3) + \dots}{i\%}, \quad (\text{A-6})$$

where, the terms in the numerator represent the numerators of the respective “Rules.”

The number of years for money to increase 20 times at an interest rate of 10% would be approximated by separating 20 into the factors 5, 2, and 2. Since $20 = 5 \times 2 \times 2$, from Equation A-6

$$n \approx \frac{167 + 72 + 72}{10} \approx 31.10 \text{ years.}$$

The exact number of years is 31.43. If m is a decimal, it may be converted to a fraction and Equation A-6 would be valid. For example if $m = 1.5$, this is the fraction $3 / 2$ and from the principles of logarithms, Equation A-6 would give $n \approx \frac{M(3) - M(2)}{i\%}$.

Trowbridge states that the “Magic Number” should be increased or decreased by 1% for each 2% that the interest rate is greater than or less than 8%, respectively, and Equation A-5 would be modified to

$$n = [1 + 0.5 \times (i - 0.08)] \frac{M(m)}{i\%}. \quad (\text{A-7})$$

However as shown in Table 1, the maximum error in the approximation is less than 3% for “reasonable” interest rates between 2% and 14%. However, for those who would like to maintain as much accuracy as

possible the rules may be modified. Using a multiple of 5 at 24% as an example, the Rule of 167 gives $n \approx 6.958$ years. The modified rule gives $n \approx 7.515$ years. The exact number is determined as $n = 7.481$ years.

Equation A-6 may be used to develop a table of composite magic numbers as shown in Table 2. This table need not be memorized. Only multiples of 2, 3 and 5 with their magic numbers 72, 114, and 167 need to be memorized, most of the other multiples can be derived from these. Unfortunately 7, 11, and 13 are “mavericks” because they are prime numbers and cannot be factored into combinations of whole numbers. Because 7 is in the range of acceptable accuracy of the Rule of 72, it might be beneficial if the magic number, 202, which is associated with it be memorized also. However, the magic number for 7 is almost the average of the numbers for 6 and 8; the magic number for 11 is almost the average of 10 and 12; and the magic number for 13 is almost the average of 12 and 14. Since all of this gives only approximations, using the averages should suffice for 7, 11, and 13 and multiples thereof.

The compound interest formulas for the types of annuities that are considered are:

For the present value of a level annuity of \$ p per year.

$$A = p \left[\frac{1 - (1 + i)^{-n}}{i} \right]; \quad (\text{A-8})$$

For the future amount of a level annuity of \$ p per year.

$$S = p \left[\frac{(1 + i)^n - 1}{i} \right]; \quad (\text{A-9})$$

For the present value of a geometrically increasing annuity beginning with \$ p_1 and increasing by $g\%$ per year

$$A = p_1 \left[\frac{1 - \left(\frac{1 + g}{1 + i} \right)^n}{i - g} \right] \quad (\text{A-10})$$

The respective numbers of years become, from equation (A-8)

$$n = - \frac{\ln \left(1 - \frac{A \times i}{p} \right)}{\ln(1 + i)}, \quad p > A \times i \quad (\text{A-11})$$

from equation A-9

$$n = \frac{\ln \left(1 + \frac{S \times i}{p} \right)}{\ln(1 + i)}, \quad (\text{A-12})$$

and from equation A-10

$$n = \frac{\ln \left(1 - \frac{A \times (i - g)}{p_1} \right)}{\ln \left(\frac{1 + g}{1 + i} \right)}, \quad p_1 > A \times (i - g) \quad (\text{A-13})$$

In order to determine the level annual payments in equations A-8 and A-9 or the initial payment in equation A-10, the present value or future amount is simply divided by the value of the calculation within the brackets.

The foregoing is valid for annuities where payments occur at the end of interest conversion periods. For annuities where the payments occur at the beginning of interest conversion periods, the p in all the above is replaced by $p \times (1 + i)$.

The future amount of a geometrically increasing annuity is given by multiplying Equation A-10 by $(1+i)^n$ and becomes

$$S = p_1 \left[\frac{(1+i)^n - (1+g)^n}{i - g} \right] \quad (\text{A-14})$$

but, the factor $(1+i)^n$ cannot be isolated in order to use any of the rules.

The final payment (savings/withdrawals) of a geometrically varying annuity of n payments may be determined by

$$p_n = p_1 \times (1+g)^{n-1} \quad (\text{A-15})$$

Table 1. Number of Years by Magic Numbers					
Rule of 72--Double					
	At Rule	6% Low	6% High		
Rate (%)	72	69.8	74.2	Exact	Error*
2	36.0	34.9		35.0	2.77%
8	9.0			9.0	0.00%
14	5.1		5.3	5.3	-2.86%
Rule of 114--Triple					
	At Rule	6% Low	6% High		
Rate (%)	114	110.6	117.4	Exact	Error*
2	57.0	55.3		55.5	2.67%
8	14.3			14.3	0.00%
14	8.1		8.4	8.4	-2.97%
Rule of 167—Quintuple					
	At Rule	6% Low	6% High		
Rate (%)	167	162.0	172.0	Exact	Error*
2	83.5	81.0		81.3	2.67%
8	20.9			20.9	0.00%
14	11.9		12.3	12.3	-2.97%

$$*\text{Error} = 100 \times \frac{\text{Rule} - \text{Exact}}{\text{Rule}}$$

Table 2--Composite Magic Numbers			
Multiple	Magic Number Expansion		Magic Number
2	2		72
3	3		114
4	2 x 2	72+72	144
5	5		167
6	2 x 3	72+114	186
7	7	$\approx (186 + 216) / 2$	202
8	2 x 2 x 2	72+72+72	216
9	3 x 3	114+114	228
10	2 x 5	72 +167	239
11	11	$\approx (239 + 258) / 2$	249
12	2 x 2 x 3	72 + 72 + 114	258
13	13	$\approx (258 + 274) / 2$	267
14	2 x 7	72 + 202	274
15	3 x 5	114 + 167	281